ON SPECIAL PIECES, THE SPRINGER CORRESPONDENCE, AND UNIPOTENT CHARACTERS

PRAMOD N. ACHAR AND DANIEL S. SAGE

Abstract. Let G be a connected reductive algebraic group over the algebraic closure of a finite field $\mathbb{F}_q$ of good characteristic. In this paper, we demonstrate a remarkable compatibility between the Springer correspondence for G and the parametrization of unipotent characters of $G(\mathbb{F}_q)$. In particular, we show that in a suitable sense, "large" portions of these two assignments in fact coincide. This extends earlier work of Lusztig on Springer representations within special pieces of the unipotent variety.

1. Introduction

Let G be a connected reductive algebraic group over the algebraic closure of a finite field $\mathbb{F}_q$ of good characteristic, and let W be its Weyl group. The Springer correspondence, which we denote by ν, assigns to each irreducible representation of W an irreducible equivariant local system on a unipotent class of G. On the other hand, part of Lusztig's parametrization of unipotent characters of finite reductive groups (or of his parametrization of unipotent character sheaves) is a map κ assigning to each irreducible representation of W an element of a certain finite set, whose members may be thought of as irreducible equivariant local systems on certain finite groups.

The main goal of this paper is demonstrate a certain compatibility between these two maps. Both maps assign to each representation of W an orbit in some space and a representation of an isotropy group. Our main result asserts that if we restrict our attention to a given two-sided cell (in a sense to be made precise later), then there is a natural order-preserving bijection between the two sets of orbits, and that the relevant isotropy groups are canonically isomorphic. Furthermore, we show that if we then identify corresponding orbits and isotropy groups, then the Springer correspondence and Lusztig's map κ actually coincide.

This theorem generalizes various results due to Lusztig [9]. For exceptional groups, its proof amounts to poring over tables from [3]. We will indicate in Section 2 exactly what poring is to be done, but we will not tabulate the results here. (Some of the calculations required for the exceptional groups case were first carried out by Kottwitz.) The proof for classical groups is carried out in Sections 3 and 4. In Section 3, we discuss various general phenomena in the classical groups and produce an outline of a proof that is independent of type, but of course lacking in combinatorial details. Section 4 supplies these details in full for type C and somewhat more sparsely in types B and D.

The research of the first author was partially supported by NSF grant DMS-0500873.
The research of the second author was partially supported by NSF grant DMS-0606300.

It has often been the case that 'combinatorial coincidences' in representation theory hint at deep underlying relationships between superficially disparate phenomena, and one wonders whether this is the case here. Since the Springer correspondence is intimately related to the theory of character sheaves, which enjoy a classification similar to that of characters of finite reductive groups, perhaps there is a purely geometric proof of the results of this paper in the character sheaf setting. (A small hint about the geometry in the problem appears in Lusztig's paper [9]: he comments on the relationship between the main results of that paper and a theorem of Kraft–Procesi [4] on the geometry of special pieces in the classical groups.) Such a proof would certainly shed light on why the results of this paper are true; perhaps it would also shed light on other aspects of the relationship among character sheaves, unipotent characters, and two-sided cells.

We are grateful to Robert Kottwitz for several valuable discussions as well as for sharing his exceptional groups calculations with us.

2. Background and definitions

2.1. Unipotent classes and Lusztig's canonical quotient group. Let $\mathcal{U}$ denote the unipotent variety in G. For any G-stable subvariety Z of $\mathcal{U}$, let $\mathcal{O}(Z)$ be the set of unipotent classes contained in Z. For each class $C \in \mathcal{O}(\mathcal{U})$, we denote by $A(C)$ the component group of the centralizer of some point $u \in C$, that is, $A(C) = G^u/(G^u)^\circ$. (Different choices of u clearly result in isomorphic groups $A(C)$; moreover, the isomorphism is determined up to an inner automorphism.) With Z as above, let

$$X(Z) = \{(C, \rho) \mid C \in \mathcal{O}(Z) \text{ and } \rho \in \operatorname{Irr}(A(C))\}.$$

Of course, this set can be identified with the set of irreducible G-equivariant local systems on G-orbits in Z.

In particular, we may regard the Springer correspondence as a map $\nu : \operatorname{Irr}(W) \to X(\mathcal{U})$. If (C, ρ) is in the image of ν, we write $\chi_{C,\rho}$ for its preimage.

Next, for a given unipotent class C, let

$$K = \bigcap_{\substack{\rho \in \operatorname{Irr}(A(C)) \\ \chi_{C,\rho} \text{ and } \chi_{C,1} \text{ are in the} \\ \text{same two-sided cell}}} \ker \rho \qquad \text{and} \qquad \bar{A}(C) = A(C)/K.$$

This group is called *Lusztig's canonical quotient* of $A(C)$. We further define

$$X^0(Z) = \{(C, \rho) \mid C \in \mathcal{O}(Z) \text{ and } \rho \in \operatorname{Irr}(\bar{A}(C))\}.$$

This set can be identified with a subset of $X(Z)$, by pulling back representations of $\bar{A}(C)$ to $A(C)$.

2.2. Special pieces. A *special piece* is, by definition, the union of a special unipotent class C_0 and those unipotent classes C in its closure with the property that for any other special class $C_1 \subsetneq \overline{C_0}$, we have $C \not\subset \overline{C_1}$. By a result of Spaltenstein [11], the full unipotent variety is the disjoint union of the various special pieces. Evidently, there is a one-to-one correspondence between special pieces and two-sided cells of W: given a two-sided cell $\mathbf{c}$, we denote by $P_{\mathbf{c}}$ the special piece determined by the special unipotent class C_0 with the property that $\chi_{C_0,1}$ is the unique special character in $\mathbf{c}$.

Let $\mathrm{Irr}(W)_{\mathbf{c}}$ denote the set of irreducible characters of W belonging to $\mathbf{c}$. (This set is a "family" of characters as defined in [6, §4.2].) Next, let

$$\mathrm{Irr}(W)^0_{\mathbf{c}} = \{\chi \in \mathrm{Irr}(W)_{\mathbf{c}} \mid \nu(\chi) = (C, \rho) \text{ where } C \subset P_{\mathbf{c}}\}.$$

In other words, $\mathrm{Irr}(W)^0_{\mathbf{c}}$ contains those characters in $\mathbf{c}$ that are sent by ν to the "correct" special piece. We also let $\mathrm{Irr}(W)^0_{\mathbf{c},C}$ denote the set of characters in $\mathrm{Irr}(W)^0_{\mathbf{c}}$ corresponding to local systems on C. Lusztig has shown that a given unipotent class C is contained in $P_{\mathbf{c}}$ if and only if $\chi_{C,1} \in \mathrm{Irr}(W)_{\mathbf{c}}$.[9] Thus, if $\chi = \chi_{C,\rho} \in \mathrm{Irr}(W)^0_{\mathbf{c}}$, then $\chi_{C,\rho}$ and $\chi_{C,1}$ both belong to $\mathbf{c}$, so ρ descends to a representation of $\bar{A}(C)$ by its definition as a quotient of $A(C)$. We conclude that if $\chi \in \mathrm{Irr}(W)^0_{\mathbf{c}}$, then $\nu(\chi) \in X^0(P_{\mathbf{c}})$. (We will see later that the converse is also true, namely that if $(C, \rho) \in X^0(P_{\mathbf{c}})$ is in the image of the Springer correspondence, then $\chi_{C,\rho} \in \mathrm{Irr}(W)^0_{\mathbf{c}}$.)

2.3. **The parametrization of unipotent characters and character sheaves.** Let $\mathbf{c}$ be a two-sided cell of W. In [6], Lusztig defined a certain finite group $\mathcal{G}_{\mathbf{c}}$ attached to $\mathbf{c}$. (In the notation of [6], this group is called $\mathcal{G}_{\mathcal{F}}$, where $\mathcal{F}$ is a family of characters.) Let $Z \subset \mathcal{G}_{\mathbf{c}}$ be a subset that is stable under conjugation. By analogy with the unipotent variety of G, we define $\mathcal{O}(Z)$ to be the set of $\mathcal{G}_{\mathbf{c}}$-orbits (i.e., conjugacy classes) in Z, and for any $y \in \mathcal{O}(\mathcal{G}_{\mathbf{c}})$, we define $I(y)$ to be the centralizer $\mathcal{G}^x_{\mathbf{c}} \subset \mathcal{G}_{\mathbf{c}}$ of some $x \in y$. (Again, $I(y)$ is uniquely determined up to inner automorphism). We further define

$$M(Z) = \{(y, \sigma) \mid y \in \mathcal{O}(Z) \text{ and } \sigma \in \mathrm{Irr}(I(y))\}.$$

Regarding Z as a discrete topological space, we may think of $M(Z)$ as the set of isomorphism classes of irreducible $\mathcal{G}_{\mathbf{c}}$-equivariant coherent sheaves on orbits in Z.

If $Z = \mathcal{G}_{\mathbf{c}}$, then the set $M(\mathcal{G}_{\mathbf{c}})$ that we have described coincides with the set $\mathcal{M}(\mathcal{G}_{\mathcal{F}})$ introduced by Lusztig in [6]. He also constructed there an explicit embedding of the set of irreducible representations in $\mathbf{c}$ into $\mathcal{M}(\mathcal{G}_{\mathbf{c}})$. We denote this map by

$$\kappa : \mathrm{Irr}(W)_{\mathbf{c}} \hookrightarrow M(\mathcal{G}_{\mathbf{c}}).$$

In the case where G is split, the set of all unipotent representations of $G(\mathbb{F}_q)$ is parametrized by the disjoint union of all $M(\mathcal{G}_{\mathbf{c}})$ as $\mathbf{c}$ ranges over two-sided cells of W. On the other hand, those unipotent representations arising by induction of the trivial character of a rational Borel subgroup are parametrized by irreducible representations of W. The map κ gives the relationship between these two parametrizations.

The same map κ also arises in the parametrization of unipotent character sheaves on G (see [8, §17.5]). Thus, the main result of the paper may be interpreted in this setting as well.

2.4. **A topology on $\bar{A}(C)$ and $\mathcal{G}_{\mathbf{c}}$.** We now fix a unipotent class C. In [1], it was shown that the set of conjugacy classes of $\bar{A}(C)$ admits a natural partial order (in which the conjugacy class of the identity is the minimal element); moreover, $\bar{A}(C)$ has a canonical structure as a Coxeter group up to conjugacy and, therefore, a well-defined class of "parabolic subgroups." (It is well known that the set of simple reflections of any irreducible finite Coxeter group is unique up to conjugacy. The point here is that $\bar{A}(C)$ is not, in general, irreducible, but it nevertheless has a

canonical decomposition into irreducible factors.) To any element $x \in \bar{A}(C)$, we associate a parabolic subgroup H_x via

$$H_x = \text{the smallest parabolic subgroup of } \bar{A}(C) \text{ containing } x.$$

(This definition makes sense because the set of parabolic subgroups is closed under intersection, so H_x is the intersection of all parabolic subgroups containing x.) Evidently, if x_1 and x_2 are conjugate, then so are H_{x_1} and H_{x_2}.

We endow $\bar{A}(C)$ with a topology by declaring a set to be open if it is a union of parabolic subgroups. In this topology, we see that for any two elements $x_1, x_2 \in \bar{A}(C)$, the conjugacy class of x_1 is contained in the closure of the conjugacy class of x_2 if and only if H_{x_2} is conjugate to a subgroup of H_{x_1}. In particular, the singleton consisting of the identity element is a dense open set.

Next, given a two-sided cell $\mathbf{c}$, recall from [6] that the finite group $\mathcal{G}_{\mathbf{c}}$ can be identified with $\bar{A}(C_{\mathbf{c}})$, where $C_{\mathbf{c}}$ is the unique special class in $P_{\mathbf{c}}$. We endow $\mathcal{G}_{\mathbf{c}}$ with a topology via this identification, and we say that a $\mathcal{G}_{\mathbf{c}}$-equivariant sheaf $\mathcal{F}$ on $\mathcal{G}_{\mathbf{c}}$ is *locally trivial* if, for each parabolic subgroup $H \subset \mathcal{G}_{\mathbf{c}}$, the action of H on $\mathcal{F}(H)$ is trivial. The stalk at any point x of a locally trivial $\mathcal{G}_{\mathbf{c}}$-equivariant sheaf then carries an action of $\mathcal{G}_{\mathbf{c}}^x/(H_x \cap \mathcal{G}_{\mathbf{c}}^x)$.

For any conjugacy class $y \in \mathcal{O}(\mathcal{G}_{\mathbf{c}})$, we define

$$\bar{I}(y) = \mathcal{G}_{\mathbf{c}}^x/(H_x \cap \mathcal{G}_{\mathbf{c}}^x) \simeq N(H_x)/H_x \qquad \text{for some } x \in y;$$

as with $I(y)$, this group is determined independently of the choice of x up to inner automorphism. Given a $\mathcal{G}_{\mathbf{c}}$-stable subset $Z \subset \mathcal{G}_{\mathbf{c}}$, the set of isomorphism classes of irreducible locally trivial $\mathcal{G}_{\mathbf{c}}$-equivariant sheaves on Z is parametrized by the set

$$M^0(Z) = \{(y, \sigma) \mid y \in \mathcal{O}(Z) \text{ and } \sigma \in \mathrm{Irr}(\bar{I}(y))\}.$$

2.5. **Statement of the main result.** We regard the set $\mathcal{O}(P_{\mathbf{c}})$ as being partially ordered in the usual way: $C_1 \leq C_2$ if $C_1 \subset \overline{C_2}$. In addition, we also endow $\mathcal{O}(\mathcal{G}_{\mathbf{c}})$ with an analogous partial order, using the topology defined above.

Theorem 2.1. *Let $\mathbf{c}$ be a two-sided cell in W.*

(1) *There is an order-preserving injective map $t : \mathcal{O}(P_{\mathbf{c}}) \to \mathcal{O}(\mathcal{G}_{\mathbf{c}})$, characterized by the property that $(t(C), 1) = \kappa(\chi_{C,1})$. Moreover, the image of t consists of the set of conjugacy classes of a subgroup $\mathcal{G}'_{\mathbf{c}} \subset \mathcal{G}_{\mathbf{c}}$.*
(2) *For each $C \in \mathcal{O}(P_{\mathbf{c}})$, there is a canonical isomorphism $\bar{A}(C) \simeq \bar{I}(t(C))$, and hence a canonical bijection $T : X^0(P_{\mathbf{c}}) \to M^0(\mathcal{G}'_{\mathbf{c}})$.*
(3) *The following diagram commutes:*

$$\begin{tikzcd}[column sep=small] & \mathrm{Irr}(W)^0_{\mathbf{c}} \arrow[dl, hook, "\nu"'] \arrow[dr, hook', "\kappa"] & \\ X^0(P_{\mathbf{c}}) \arrow[rr, "\sim", "T"'] & & M^0(\mathcal{G}'_{\mathbf{c}}) \end{tikzcd}$$

(4) *In each simple type except G_2, F_4, and E_8, the maps ν and κ in the diagram are bijections for each two-sided cell. For the three remaining types, there is exactly one cell for which the maps are not surjective, corresponding to the special unipotent classes $G_2(a_1)$, $F_4(a_3)$, and $E_8(a_7)$ (using the notation of [3]). In each case, a single local system is missing from the image of ν, namely the special class together with the sign representation of $\bar{A}(C)$ (which is S_3, S_4, and S_5 respectively).*

It suffices to prove the theorem for each simple root system. Moreover, there are no nontrivial local systems in type A, so the theorem is trivial in this case.

Proof for the exceptional types. Parts (1) and (2) of this theorem were already proved in Theorem 0.4 and Proposition 0.7 of [9]. (Proposition 0.7(b) of *loc. cit.* asserts that the group we have called $\bar{I}(t(C))$ (there, $N(H_C)/H_C$) is isomorphic to $A(C)$, not $\bar{A}(C)$. However, that statement is only meant to apply to *nonspecial* classes in the exceptional groups. By comparing the tables in [10] with those in [3], one verifies that for all such classes, it is in fact the case that $A(C) = \bar{A}(C)$.)

Only the commutativity of the diagram in part (3) and the statements about the images of the maps ν and κ in part (4) remain. For this, one undertakes the laborious task of comparing tables for the Springer correspondence in, say, [3] with Lusztig's enumeration of values of κ in [6]. □

Remarks 2.2. (1) The commutative diagram in part (3) of the theorem restricts to give a diagram for each unipotent orbit $C \subset P_{\mathbf{c}}$:

$$\begin{tikzcd} & \mathrm{Irr}(W)^0_{\mathbf{c},C} \arrow[dl, hook', "\nu"'] \arrow[dr, hook, "\kappa"] & \\ X^0(C) \arrow[rr, "\sim", "T"'] & & M^0(t(C)) \end{tikzcd}$$

Again, the maps ν and κ are bijections except in the three cases mentioned above.

(2) It is a corollary of the theorem and its proof that for any $\chi \in \mathrm{Irr}(W)$, $\chi \in \mathrm{Irr}(W)^0_{\mathbf{c}}$ if and only if $\nu(\chi) \in X^0(P_{\mathbf{c}})$. Indeed, this follows from part (4) of the theorem together with the injectivity of the Springer correspondence for all cells besides the three anomalies. However, in these three cases, the single element in $X^0(P_{\mathbf{c}})$ not in $\nu(\mathrm{Irr}(W)^0_{\mathbf{c}})$ is not in the image of the Springer correspondence.

3. The classical types: Outline

The proof of the main theorem for the classical types B, C, and D comes down to calculations with various combinatorial objects that parametrize $\mathrm{Irr}(W)$, $X(\mathcal{U})$, and $X(\mathcal{G}_{\mathbf{c}})$. The details differ in the three types, but they can all be placed in a common framework, which we will now develop.

3.1. $\bar{A}(C)$ **for classical groups.** Recall that for the classical root systems, the groups $\mathcal{G}_{\mathbf{c}}$ are always products of copies of $\mathbb{Z}/2\mathbb{Z}$. In particular, they are always abelian. Thus, the set $\mathrm{Irr}(\mathcal{G}_{\mathbf{c}})$ can be identified with the group $\hat{\mathcal{G}}_{\mathbf{c}} = \mathrm{Hom}(\mathcal{G}_{\mathbf{c}}, \mathbb{C}^\times)$, and $M(\mathcal{G}_{\mathbf{c}})$ can, in turn, be identified with $\mathcal{G}_{\mathbf{c}} \times \hat{\mathcal{G}}_{\mathbf{c}}$. In this section, we will give a particular realization of these spaces that is well-suited to the combinatorics that will follow. Furthermore, we describe the sets $\mathrm{Irr}(\mathcal{G}_{\mathbf{c}}/H_x)$ with respect to this realization.

Suppose $\mathcal{G}_{\mathbf{c}} \simeq (\mathbb{Z}/2\mathbb{Z})^f$. Let $\tilde{V}_{\mathbf{c}}$ be the $(2f+1)$-dimensional $\mathbb{F}_2$-vector space with basis

$$e_0, e_1, \ldots, e_{2f}$$

and $V_{\mathbf{c}}$ its quotient by the relation

$$e_0 + e_2 + e_4 + \cdots + e_{2f} = 0 \tag{1}$$

(so $\dim V_{\mathbf{c}} = 2f$). We endow $\tilde{V}_{\mathbf{c}}$ with a symmetric bilinear form given by

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This form has kernel spanned by $e_0 + e_2 + \cdots + e_{2f}$, so it induces a nondegenerate symmetric bilinear form on $V_{\mathbf{c}}$. We make the identification

$$\mathcal{G}_{\mathbf{c}} = \operatorname{span}\{e_1, e_3, \ldots, e_{2f-1}\},$$

and we remark, in particular, that the canonical Coxeter generators referred to in Section 2.4 are precisely the elements $e_1, e_3, \ldots, e_{2f-1}$. Via the bilinear form $\langle , \rangle$, we can also identify

$$\hat{\mathcal{G}}_{\mathbf{c}} = \operatorname{Hom}(\mathcal{G}_{\mathbf{c}}, \mathbb{C}^\times) = \operatorname{span}\{e_0, e_2, \ldots, e_{2f}\},$$

viewed as a subspace of $V_{\mathbf{c}}$. Note that every character has exactly two representatives in $\tilde{V}_{\mathbf{c}}$.

Now, take $x \in \mathcal{G}_{\mathbf{c}}$. There is a set of integers $n(x) \subset \{1, 3 \ldots, 2f-1\}$ such that

$$x = \sum_{j \in n(x)} e_j.$$

Evidently, the parabolic subgroup H_x is simply the span of $\{e_i \mid i \in n(x)\}$. To describe $\operatorname{Irr}(\mathcal{G}_{\mathbf{c}}/H_x)$ as a subset of $\hat{\mathcal{G}}_{\mathbf{c}}$, we need a slightly refined description of $n(x)$. An element $\lambda \in \operatorname{span}\{e_0, e_2, \ldots, e_{2f}\}$ is called a *block* of x if it is of the form

$$\lambda = e_i + e_{i+2} + \cdots + e_{i+2k}$$

where

$$\{i+1, i+3, \ldots, i+2k-1\} \subset n(x), \quad i-1 \notin n(x), \quad i+2k+1 \notin n(x).$$

Note that i must be even here. It is permitted that $k = 0$, so a single term e_i is a block if i is even and $i-1, i+1 \notin n(x)$. We will also view λ as a character in $\hat{\mathcal{G}}_{\mathbf{c}}$ by taking its image in $V_{\mathbf{c}}$. Next, the element $m \in \mathcal{G}_{\mathbf{c}}$ given by

$$m = e_{i+1} + e_{i+3} + \cdots + e_{i+2k-1}$$

is called the *component* of x associated to the block λ. (If x has a block that is a single term e_i, then the associated component is 0.)

Clearly, x is the sum of its components, with distinct components orthogonal and each e_j for j odd appearing in at most one component. Similarly, every e_j with j even appears in exactly one block of x. Thus, the sum of the blocks is 0 (by equation (1)), and any proper subset of the blocks is linearly independent.

Lemma 3.1. $\operatorname{Irr}(\mathcal{G}_{\mathbf{c}}/H_x)$ *is the subspace of* $\hat{\mathcal{G}}_{\mathbf{c}}$ *spanned by the images of the blocks of* x.

Proof. Let λ be a block of x, and let m be its associated component. It is clear that $\langle \lambda, e_j \rangle = 0$ for each e_j appearing in m. Moreover, neither $i-1$ nor $i+2k+1$ belongs to $n(x)$, so in fact, $\langle \lambda, e_j \rangle = 0$ for all $j \in n(x)$. Thus, $\lambda|_{H_x}$ is indeed trivial.

Conversely, let $\mu \in \hat{\mathcal{G}}_{\mathbf{c}}$ be such that some, but not all, of the e_j appearing in λ appear in μ. (This is independent of the choice of representative of μ in $\tilde{V}_{\mathbf{c}}$.) In other words, μ is not in the span of the blocks of x. This means that we can find a pair of consecutive terms in λ, say e_{2p} and e_{2p+2}, such that exactly one of them appears in μ. Then $2p+1 \in n(x)$, yet it is clear that $\langle \mu, e_{2p+1} \rangle = 1$. Thus, $\mu|_{H_x} \neq 0$. $\square$

3.2. **Symbols and u-symbols.** The combinatorial objects with which all our calculations in the classical types will be carried out are called *symbols* and *u-symbols*. Both of these are certain finite sequences $(a_0, a_1, \ldots, a_r)$ of nonnegative integers, satisfying various additional conditions (that are different for symbols and for u-symbols). The value of r (*i.e.*, the length of the sequences), as well as the precise form of the additional conditions, depend on the type. The set of symbols and the set of u-symbols both parameterize $\mathrm{Irr}(W)$ explicitly. Accordingly, there is a bijection

$$i : \{u\text{-symbols}\} \to \{\text{symbols}\},$$

which maps the u-symbol of an irreducible representation to its symbol.

Remark 3.2. Our usage of the terms "symbol" and "u-symbol" is close to that of [9], but narrower than the usage elsewhere in the literature. The definitions that we will give correspond to what other authors refer to as symbols or u-symbols "of defect 1" (for types B and C) or "of defect 0" (for type D).

Also, symbols and u-symbols are usually written as arrays consisting of two rows of numbers. Here, following [9], we will write them simply as finite sequences, in which the even-numbered entries $a_0, a_2, \ldots$ may be thought of as the "upper row," and the odd-numbered entries as the "lower row." In all three types, symbols will have strictly increasing upper and lower rows while u-symbols will have $a_{i+2} - a_i \geq 2$.

Symbols and u-symbols both fall into families determined by their entries.

Definition 3.3. Two symbols $\mathbf{a}$ and $\mathbf{b}$ are said to be *congruent* if they contain the same entries (with the same multiplicities), possibly in different orders. A symbol $(a_0, \ldots, a_r)$ is *special* if $a_0 \leq \cdots \leq a_r$.

Two u-symbols $\mathbf{a}$ and $\mathbf{b}$ are said to be *similar* if they contain the same entries (with the same multiplicities), possibly in different orders. A u-symbol $(a_0, \ldots, a_r)$ is *distinguished* if $a_0 \leq \cdots \leq a_r$.

In particular, each congruence class of symbols contains a unique special symbol [9, Lemmas 1.6, 2.6, 3.6], and each similarity class of u-symbols contains a unique distinguished symbol [7, §11.5].

Many of the combinatorial results we require will be easier to state and prove if we work with the indices of entries in a symbol or u-symbol rather than the entries themselves. In other words, we make statements in terms of subsets of $\{0, \ldots, r\}$ rather than subsequences of $(a_0, \ldots, a_r)$. The following notation will be particularly useful:

$$[k, l] = \{k, k+1, \ldots, l\}.$$

Definition 3.4. Let $\mathbf{a} = (a_0, \ldots, a_r)$ be a u-symbol, and let $\mathbf{a}' = i(\mathbf{a}) = (a'_0, \ldots, a'_r)$ be the corresponding symbol.

A number $k \in [0, r]$ is called an *isolated point* (of $\mathbf{a}$ or $\mathbf{a}'$) if $a'_k \neq a'_l$ for all $l \neq k$.

A subset $[k, l] \subset [0, r]$ is called a *ladder* of $\mathbf{a}$ if, for some a, we have

$$(a_k, a_{k+1}, \ldots, a_l) = (a, a+1, \ldots, a+l-k),$$
$$a_{k-1} < a - 1 \text{ if } k > 0, \qquad \text{and} \qquad a_{l+1} > a + l - k + 1 \text{ if } l < r.$$

(Ladders are called intervals in some earlier references.)

A subset $[k, l] \subset [0, r]$ is called a *staircase* of $\mathbf{a}$ if, for some a, we have

$$(a_k, a_{k+1}, \ldots, a_l) = (a, a, a+2, a+2, \ldots, a+l-k-1, a+l-k-1),$$
$$a_{k-2} < a-2 \text{ if } k > 1, \qquad \text{and} \qquad a_{l+2} > a+l-k+1 \text{ if } l < r-1.$$

(Note that $l - k + 1$ is necessarily even, if $[k, l]$ is a staircase.)

A set $[k, l]$ that is either a ladder or a staircase is called a *part*. The number k is called the *bottom* of the part, and l is called the *top*. The *length* of the part is defined to be $l - k + 1$.

We will only be interested in parts for distinguished u-symbols. It is easy to see that the set of indices for any finite nondecreasing sequence of integers satisfying $a_{i+2} - a_i \geq 2$ is the disjoint union of ladders and staircases. In particular, for a distinguished u-symbol, the full set of indices $[0, r]$ is the disjoint union of all the parts.

We now describe a construction that will have multiple uses in the sequel.

Definition 3.5. Let $\mathbf{a} = (a_0, \ldots, a_r)$ be a symbol (resp. u-symbol), and let μ be a subset of $\{0, \ldots, r\}$, such that for each $i \in \mu$, the entry a_i is distinct from all other entries of $\mathbf{a}$. Let $A = \{a_0, a_2, a_4, \ldots\}$, $B = \{a_1, a_3, a_5, \ldots\}$, and $Z = \{a_i \mid i \in \mu\}$. Define two new sequences as follows:

$$A' = (b_0, b_2, b_4 \ldots) = (A \smallsetminus Z) \cup (B \cap Z), \text{ arranged in increasing order,}$$
$$B' = (b_1, b_3, b_5 \ldots) = (B \smallsetminus Z) \cup (A \cap Z), \text{ arranged in increasing order.}$$

If the lengths of A' and B' are the same as the lengths of A and B, respectively, then we can combine them into a single sequence $\mathbf{b} = (b_0, b_1, b_2, \ldots, b_r)$, which is readily seen (for each type) to be a new symbol (resp. u-symbol). We say that $\mathbf{b}$ is obtained by *twisting* $\mathbf{a}$ by μ, and we denote it by $\mathbf{b} = \mathbf{a}^\mu$.

It is immediate that the twisting operation preserves congruence classes of symbols and similarity classes of u-symbols.

3.3. **Symbols, cells, and κ.** Following [5], we know that two symbols are congruent if and only if they correspond to representations of W lying in the same two-sided cell. Moreover, the symbols we have called "special" are precisely those corresponding to special representations of W.

Now, note that any two congruent symbols have the same set of entries at isolated points, so there is a natural bijection between their respective sets of isolated points.

Definition 3.6. Let $\mathbf{a}_0$ be a special symbol, and let $\mathbf{a}$ be a symbol in its congruence class. An isolated point of $\mathbf{a}$ is said to be *displaced* if it has parity opposite to that of the corresponding isolated point of $\mathbf{a}_0$. An isolated point of a u-symbol $\mathbf{b}$ is displaced if it is displaced in $i(\mathbf{b})$. Every symbol (and u-symbol) has an even number of displaced isolated points.

Later, for each classical type, we will obtain a more direct characterization of displaced isolated points.

Let $\tilde{V}(\mathbf{a}_0)$ be the power set of the set of isolated points, regarded as an $\mathbb{F}_2$-vector space with addition given by symmetric difference of sets. Let $V(\mathbf{a}_0)$ be the subspace of $\tilde{V}(\mathbf{a}_0)$ consisting of sets with even cardinality. We endow $V(\mathbf{a}_0)$ with a symmetric bilinear form by putting $\langle v, w\rangle = |v \cap w| \mod 2$. (Here $|v \cap w|$ is the cardinality of the set $v \cap w$.)

Since any symbol has an even number of displaced isolated points, there is a natural map

$$\tilde{\kappa} : \left\{ \begin{array}{c} \text{symbols} \\ \text{congruent to } \mathbf{a}_0 \end{array} \right\} \to V(\mathbf{a}_0) \qquad \text{via} \qquad \mathbf{a} \mapsto \left\{ \begin{array}{c} \text{displaced} \\ \text{isolated points of } \mathbf{a} \end{array} \right\}$$

Lusztig has shown that if $\mathbf{c}$ is the two-sided cell corresponding to the special symbol $\mathbf{a}_0$, then there is a natural surjective map π from $V(\mathbf{a}_0)$ to the space $V_{\mathbf{c}}$ of Section 3.1; in fact, it is an isomorphism (even an isometry) except in type D. [6] We will describe this map explicitly in each type by labelling certain elements of $V(\mathbf{a}_0)$ with the names $e_0, e_1, \ldots, e_{2f}$. With this in mind, the map $\tilde{\kappa}$ defined above can be regarded as a combinatorial version of Lusztig's map $\kappa : \mathrm{Irr}(W)_{\mathbf{c}} \to V_{\mathbf{c}} = M(\mathcal{G}_{\mathbf{c}})$, in that the following diagram commutes:

$$\begin{tikzcd}
\left\{ \begin{array}{c} \text{symbols} \\ \text{congruent to } \mathbf{a}_0 \end{array} \right\} \arrow[r, "\tilde{\kappa}"] \arrow[d, equal] & V(\mathbf{a}_0) \arrow[d, "\pi"] \\
\mathrm{Irr}(W)_{\mathbf{c}} \arrow[r, "\kappa"] & V_{\mathbf{c}}
\end{tikzcd} \tag{2}$$

(For a detailed discussion, see [6, pp. 86–88] for types B and C and [6, pp. 92–94] for type D.)

It is not difficult to see that if $\mathbf{a}$ is a symbol and μ is a set consisting of an even number of isolated points, then $\tilde{\kappa}(\mathbf{a}^\mu) = \tilde{\kappa}(\mathbf{a}) + \mu$ (assuming $\mathbf{a}^\mu$ is defined). A more significant observation is that, by the commutativity of the above diagram, we have

$$\kappa(\mathbf{a}^\mu) = \kappa(\mathbf{a}) + \pi(\mu). \tag{3}$$

3.4. **u-Symbols and the Springer correspondence.** Following [7], we know that two u-symbols are similar if and only if the Springer correspondence maps the corresponding representations to local systems on the same unipotent class. Moreover, the distinguished u-symbols are those corresponding to Springer representations (that is, representations mapped to the trivial local system on some unipotent class). There is thus a bijection between distinguished u-symbols and unipotent classes.

Given a distinguished u-symbol $\mathbf{a} = (a_0, \ldots, a_r)$, let $H(\mathbf{a})$ be the power set of the set of ladders of $\mathbf{a}$, regarded as an $\mathbb{F}_2$-vector space with addition given by symmetric difference of sets.

Let C be the unipotent class corresponding to $\mathbf{a}$. In each classical type, there is a surjective map $p : H(\mathbf{a}) \to \mathrm{Irr}(A(C))$, such that the set of all ladders lies in its kernel. (To be more precise, $\mathrm{Irr}(A(C))$ can be canonically identified with a quotient of some subspace of $H(\mathbf{a})$, as described in [3, pp. 419–423]. The map p may not be canonically determined, but we may choose it so that its kernel contains the set of all ladders.)

By abuse of language, we will often regard an element $\mu \in H(\mathbf{a})$ as a subset of $[0, r]$ rather than as a set of ladders, by replacing μ by the union of its members. This abuse allows us to speak of twisting a u-symbol by an element of $H(\mathbf{a})$.

It turns out that all u-symbols in the similarity class of $\mathbf{a}$ arise by twisting $\mathbf{a}$ by a suitable set of ladders.[7] Moreover, this set of ladders is uniquely determined. Given a u-symbol $\mathbf{a}'$ similar to $\mathbf{a}$, we denote by $\tilde{\nu}(\mathbf{a}') \in H(\mathbf{a})$ the unique element such that $\mathbf{a}' = \mathbf{a}^{\tilde{\nu}(\mathbf{a}')}$. The following analogue of (2) commutes, so $\tilde{\nu}$ can be thought

of as a combinatorial version of the Springer correspondence:

$$
\begin{tikzcd}
\left\{\begin{matrix} u\text{-symbols} \\ \text{similar to } \mathbf{a} \end{matrix}\right\} \arrow[rr, "\tilde{\nu}"] \arrow[d, equal] & & H(\mathbf{a}) \arrow[d, "p"] \\
\operatorname{Irr}(W)_C \arrow[r, "\nu"] & C \times \operatorname{Irr}(A(C)) \arrow[r, "\cong"] & \operatorname{Irr}(A(C))
\end{tikzcd} \tag{4}
$$

(A detailed account is given in [7, §12] for type C and [7, §13] for types B and D.) Here, $\operatorname{Irr}(W)_C \subset \operatorname{Irr}(W)$ is just the set of representations to which ν assigns local systems on C. In addition, we have an analogue of (3):

$$\nu(\mathbf{a}^\mu) = (C, p(\mu)), \tag{5}$$

whenever $\mathbf{a}^\mu$ is defined.

We will also introduce subsets of $[0, r]$ called the *blocks* of $\mathbf{a}$, which are certain unions of ladders and staircases. Each block will contain at least one ladder, and each ladder will be contained in a block. Moreover, distinct blocks will have empty intersection. Let $B(\mathbf{a})$ be the set of blocks and $\bar{H}(\mathbf{a})$ the power set of $B(\mathbf{a})$. By assigning to a block the set of ladders contained within it, we obtain an injective map from $\bar{H}(\mathbf{a})$ to $H(\mathbf{a})$, and we will regard $\bar{H}(\mathbf{a})$ as a subset of $H(\mathbf{a})$ via this map. In particular, twisting by a set of blocks will mean twisting by the set of ladders in these blocks.

Those u-symbols which may be obtained from $\mathbf{a}$ upon twisting by a set of blocks will be of particular interest. Let $\mathcal{T}(\mathbf{a})$ be the set of those elements $\mu \in \bar{H}(\mathbf{a})$ for which $\mathbf{a}^\mu$ is defined. We will see that it is a transversal of the two element subgroup $\{\varnothing, B(\mathbf{a})\}$ and thus a set of size $2^{|B(\mathbf{a})|-1}$ (except in the trivial case where $B(\mathbf{a})$ is empty, which can only occur in type D).

3.5. **Isolated points and blocks.** In this section, we give an outline of the proof of Theorem 2.1 in the classical root systems. Specifically, we state a number of intermediate lemmas and propositions and indicate how these statements together imply the theorem. The proofs of most of the intermediate results are slightly different in each type and depend on the details of the combinatorial definitions of symbols and u-symbols. Here, we only provide those proofs which are independent of type.

We fix the following notations: let $\mathbf{a}$ be a distinguished u-symbol, and let $\mathbf{a}_0$ be the unique u-symbol such that $i(\mathbf{a}_0)$ is special and congruent to $i(\mathbf{a})$. (Note that $\mathbf{a}_0$ is necessarily distinguished, since all special representations of W are Springer representations.) Let $\mathbf{c}$ be the two-sided cell corresponding to $i(\mathbf{a}_0)$, and let C be the unipotent class corresponding to $\mathbf{a}$.

The first result tells us where isolated points occur in a u-symbol:

Proposition 3.7. *Let* $\mathbf{a}$ *be a distinguished* u*-symbol. The top and bottom of each block of* $\mathbf{a}$ *are nondisplaced isolated points. In addition to these, each block contains an even number (possibly zero) of additional isolated points, all of which are displaced.*

We remark that the terms "top" and "bottom" as applied to blocks are not quite the same as for ladders and staircases—the details vary by type and will be given in Section 4. In particular, not every block has both a top and a bottom, so this proposition does not imply that there are necessarily an even number of isolated

points. Nevertheless, we will see in each case that there is at most one block with an odd number of isolated points.

Given a block b of a distinguished u-symbol $\mathbf{a}$, define $\lambda_b \in V(i(\mathbf{a}_0))$ by

$$\lambda_b = \begin{cases} \text{the set of all isolated points in } b & \text{if that set has even cardinality;} \\ \text{the set of all isolated points } \textit{not} \text{ in } b & \text{otherwise.} \end{cases}$$

The last sentence of the previous paragraph implies that λ_b does indeed belong to $V(i(\mathbf{a}_0))$. We extend the definition of λ_b to elements $b \in \bar{H}(\mathbf{a})$ by linearity. Next, let $m_b \in V(i(\mathbf{a}_0))$ be the set of displaced isolated points of b. (Accordingly, for most blocks, m_b is obtained from λ_b by omitting the top and bottom of b.)

Note that according to the preceding proposition, $\tilde{\kappa}(i(\mathbf{a})) = \sum m_b$, where b runs over the blocks of $\mathbf{a}$.

Proposition 3.8. *For any block b of $\mathbf{a}$, the element $\pi(m_b) \in \mathcal{G}_{\mathbf{c}} \times \hat{\mathcal{G}}_{\mathbf{c}}$ is of the form $(x, 1)$ for some $x \in \mathcal{G}_{\mathbf{c}}$.*

As a consequence of this proposition, we recover Lusztig's result [9] that $\kappa(\chi_{C,1})$ is of the form $(x, 1)$ for some $x \in \mathcal{G}_{\mathbf{c}}$. In particular, we henceforth have available to us the map $t : \mathcal{O}(P_{\mathbf{c}}) \to \mathcal{O}(\mathcal{G}_{\mathbf{c}})$ which takes the unipotent orbit $C \subset P_{\mathbf{c}}$ to the first coordinate of $\kappa(\chi_{C,1})$.

Proposition 3.9. *If C_1 and C_2 are two unipotent classes in the same special piece, then $t(C_1) \le t(C_2)$ in the partial order on $\mathcal{G}_{\mathbf{c}}$ if and only if C_1 is in the closure of C_2.*

Once this assertion is established, we will have completed the proof of part (1) of Theorem 2.1. (The fact that the image of t is a subgroup of $\mathcal{G}_{\mathbf{c}}$ is proved in Theorem 0.4 of [9].)

Proposition 3.10. *The elements $\pi(m_b)$, as b runs over the blocks of $\mathbf{a}$, are the components of $t(C)$, and the elements $\pi(\lambda_b)$ are the images in $V_{\mathbf{c}}$ of the blocks of $t(C)$.*

Since the blocks of $\mathbf{a}$ are a basis for $\bar{H}(\mathbf{a})$, and $\mathrm{Irr}(\mathcal{G}_{\mathbf{c}}/H_x)$ is generated by the images of the blocks of x, we see that the map $b \mapsto \lambda_b$ induces a surjective map $\bar{H}(\mathbf{a}) \to \mathrm{Irr}(\mathcal{G}_{\mathbf{c}}/H_{t(C)}) = \mathrm{Irr}(\bar{I}(t(C)))$. Moreover, it follows from the remark immediately preceding Lemma 3.1 that the kernel is generated by the sum of all the blocks of $\mathbf{a}$. We will show that this is also the kernel of $p|_{\bar{H}(\mathbf{a})} : \bar{H}(\mathbf{a}) \to \mathrm{Irr}(A(C))$, but first, we must examine the set of u-symbols obtainable from $\mathbf{a}$ by twisting by a set of blocks. We will see that they correspond to the characters in $\mathrm{Irr}(W)^0_{\mathbf{c},C}$. Recall that $\mathcal{T}(\mathbf{a})$ is the subset of $\bar{H}(\mathbf{a})$ consisting of those elements μ for which $\mathbf{a}^\mu$ is defined.

Lemma 3.11. *The set $\mathcal{T}(\mathbf{a})$ contains at most one element from any coset of $\{\varnothing, B(\mathbf{a})\}$.*

Proof. Suppose that both μ and $\mu^c = \mu + B(\mathbf{a})$ are in $\mathcal{T}(\mathbf{a})$. Since $B(\mathbf{a})$ is in the kernel of p, (5) implies that $\nu(\mathbf{a}^\mu) = \nu(\mathbf{a}^{\mu^c})$, and the injectivity of the Springer correspondence shows that $\mathbf{a}^\mu = \mathbf{a}^{\mu^c}$. This is a contradiction if $B(\mathbf{a})$ is nonempty, since a u-symbol $\mathbf{a}'$ similar to $\mathbf{a}$ is obtained by twisting $\mathbf{a}$ by a unique set of ladders, namely $\tilde{\nu}(\mathbf{a}')$. (The statement is trivial if $B(\mathbf{a})$ is empty.) $\square$

Proposition 3.12. *If* $\mathbf{a}$ *is a distinguished* u*-symbol, then* $i(\mathbf{a}^b) = (i(\mathbf{a}))^{\lambda_b}$ *for any* $b \in \mathcal{T}(\mathbf{a})$*. Furthermore,* $\mathcal{T}(\mathbf{a})$ *is a transversal in* $\bar{H}(\mathbf{a})$ *of the subgroup* $\{\varnothing, B(\mathbf{a})\}$*.*

We now compute the kernel and range of the map $p|_{\bar{H}(\mathbf{a})} : \bar{H}(\mathbf{a}) \to \mathrm{Irr}(A(C))$.

Proposition 3.13. *The kernel of the restriction* $p|_{\bar{H}(\mathbf{a})} : \bar{H}(\mathbf{a}) \to \mathrm{Irr}(A(C))$ *is generated by* $B(\mathbf{a})$*, the sum of all the blocks of* $\mathbf{a}$*.*

Proof. Recall that $p : H(\mathbf{a}) \to \mathrm{Irr}(A(C))$ was defined such that the set of all ladders is in its kernel. Since every ladder is in a block, it is clear that the set of all blocks is in the kernel of $p|_{\bar{H}(\mathbf{a})}$. On the other hand, distinct μ's in $\mathcal{T}(\mathbf{a})$ result in distinct twisted u-symbols, so the injectivity of the Springer correspondence and the commutativity of (4) imply that $p|_{\bar{H}(\mathbf{a})}$ must be injective when restricted to $\mathcal{T}(\mathbf{a})$. But $\mathcal{T}(\mathbf{a})$ is a transversal of $\{\varnothing, B(\mathbf{a})\}$, so the subgroup $\{\varnothing, B(\mathbf{a})\}$ is indeed the kernel of $p|_{\bar{H}(\mathbf{a})}$. $\square$

As a result, the map $p|_{\bar{H}(\mathbf{a})} : \bar{H}(\mathbf{a}) \to \mathrm{Irr}(A(C))$ factors through the surjection $\bar{H}(\mathbf{a}) \to \mathrm{Irr}(\bar{I}(t(C)))$, inducing an injective map $\mathrm{Irr}(\bar{I}(t(C))) \hookrightarrow \mathrm{Irr}(A(C))$.

Since $\bar{A}(C)$ is a quotient of $A(C)$, $\mathrm{Irr}(\bar{A}(C))$ is naturally a subset of $\mathrm{Irr}(A(C))$. In fact:

Proposition 3.14. *The image of* $p|_{\bar{H}(\mathbf{a})} : \bar{H}(\mathbf{a}) \to \mathrm{Irr}(A(C))$ *is precisely* $\mathrm{Irr}(\bar{A}(C))$*.*

An immediate consequence of this last proposition is that the image of the aforementioned injective map $\mathrm{Irr}(\bar{I}(t(C))) \hookrightarrow \mathrm{Irr}(A(C))$ is $\mathrm{Irr}(\bar{A}(C))$. Part (2) of Theorem 2.1 is thus established; the map $T : X^0(P_{\mathbf{c}}) \to M^0(\mathcal{G}'_{\mathbf{c}})$ is described by $T(C, p(b)) = (t(C), \pi(\lambda_b))$.

Proposition 3.15. *There is a bijection* $\mathcal{T}(\mathbf{a}) \to \mathrm{Irr}(W)^0_{\mathbf{c},C}$ *given by the map* $b \mapsto \mathbf{a}^b$*.*

Proof. First, suppose that $b \in \mathcal{T}(\mathbf{a})$, and consider the u-symbol $\mathbf{a}^b$. It is similar to $\mathbf{a}$ while Proposition 3.12 shows that $i(\mathbf{a}^b)$ is congruent to $i(\mathbf{a})$. This says precisely that $\mathbf{a}^b$ corresponds to a character in $\mathrm{Irr}(W)^0_{\mathbf{c},C}$.

Now suppose that $\mathbf{a}'$ is a u-symbol corresponding to a representation in $\mathrm{Irr}(W)^0_{\mathbf{c},C}$. Setting $\nu(\mathbf{a}') = (C, \rho)$, we have $\rho \in \mathrm{Irr}(\bar{A}(C))$, and since the image of $p|_{\bar{H}(\mathbf{a})} : \bar{H}(\mathbf{a}) \to \mathrm{Irr}(A(C))$ is precisely $\mathrm{Irr}(\bar{A}(C))$ (by Proposition 3.14), we can choose $b \in \bar{H}(\mathbf{a})$ with $p(\mu) = \rho$. Moreover, since $\mathcal{T}(\mathbf{a})$ is a transversal for the kernel of this map, we can choose $b \in \mathcal{T}(\mathbf{a})$. This means that $\mathbf{a}^b$ is defined, so by (5), $\nu(\mathbf{a}^b) = (C, \rho) = \nu(\mathbf{a}')$. The injectivity of ν gives $\mathbf{a}' = \mathbf{a}^b$. $\square$

We obtain the following representation-theoretic corollary:

Corollary 3.16. *The Springer correspondence* ν *restricts to bijections* $\mathrm{Irr}(W)^0_{\mathbf{c},C} \to X^0(C)$ *and* $\mathrm{Irr}(W)^0_{\mathbf{c}} \to X^0(P_{\mathbf{c}})$*.*

Proof. The fact that $\mathcal{T}(\mathbf{a})$ is a transversal for the kernel of $p|_{\bar{H}(\mathbf{a})}$ implies that $|\mathcal{T}(\mathbf{a})| = |\mathrm{Irr}(\bar{A}(C))| = |X^0(C)|$. The first bijection follows, and we get the second by taking the union over unipotent classes in $P_{\mathbf{c}}$. $\square$

We are now ready to complete the proof of the main theorem. Given $\chi = \chi_{C,\rho} \in \mathrm{Irr}(W)^0_{\mathbf{c}}$, let $\mathbf{a}'$ be its u-symbol and $\mathbf{a}$ the distinguished u-symbol for C. By Proposition 3.15, $\mathbf{a}' = \mathbf{a}^b$ for some $b \in \mathcal{T}(\mathbf{a})$. We know that $\kappa(i(\mathbf{a})) = (t(C), 1)$, so

equation (3) and Proposition 3.12 give $\kappa(i(\mathbf{a})^{\lambda_b}) = (t(C), \pi(\lambda_b))$. In other words, we have $\kappa(\chi_{C,\rho}) = (t(C), \pi(\lambda_b)) = T(C, \rho)$. Therefore,

$$\kappa(\chi) = T(\nu(\chi)) \qquad \text{for all } \chi \in \mathrm{Irr}(W)^0_{\mathbf{c}}.$$

This establishes the commutativity of the diagram in part (3) of Theorem 2.1. Finally, we obtain part (4) of the theorem: the restriction of ν in the diagram is a bijection by Proposition 3.16, and so $\kappa = T \circ \nu$ is a composition of bijections. The proof of the theorem is thus completed.

4. The classical types: Combinatorics

It remains, of course, to fill in a number of details separately in each of the classical types. The missing ingredients are as follows:

(1) Definitions of "symbol," "u-symbol," and "block."
(2) A careful study of where isolated points may occur.
(3) Proofs of Propositions 3.7–3.10, 3.12, and 3.14.

Below, we give full details for each of these steps in type C. In particular, item (2) is treated in Lemmas 4.2 and 4.3. However, types B and D receive a much more cursory treatment: we provide the requisite definitions, and we state without proof the necessary results on isolated points (see Lemmas 4.12 and 4.13 for type B, and Lemmas 4.16 and 4.17 for type D), but we make no comment at all on the other propositions, aside from giving an explicit characterization (again without proof) of the transversal $\mathcal{T}(\mathbf{a})$. All the missing proofs in types B and D are, of course, quite similar to the corresponding proofs in type C.

4.1. **Type C.** Let $\Psi'_{2n,m}$ (resp. $\Phi'_{2n,m}$) be the set of sequences $\mathbf{a} = (a_0, a_1, \ldots, a_{2m})$ satisfying the following conditions:

$$0 \le a_0, \quad 1 \le a_1, \quad a_i \le a_{i+2} - 2, \quad \sum a_i = n + 2m^2 + m \qquad \text{for } \mathbf{a} \in \Psi'_{2n,m},$$
$$0 \le a_0, \quad 0 \le a_1, \quad a_i \le a_{i+2} - 1, \quad \sum a_i = n + m^2 \qquad \text{for } \mathbf{a} \in \Phi'_{2n,m}$$

There are embeddings $S : \Psi'_{2n,m} \to \Psi'_{2n,m+1}$, $S : \Phi'_{2n,m} \to \Phi'_{2n,m+1}$ given by

$$S(\mathbf{a}) = (0, 1, a_0 + 2, \ldots, a_{2m} + 2) \qquad \text{for } \mathbf{a} \in \Psi'_{2n,m},$$
$$S(\mathbf{a}) = (0, 0, a_0 + 1, \ldots, a_{2m} + 1) \qquad \text{for } \mathbf{a} \in \Phi'_{2n,m}.$$

These maps are called *shift operations*. Put $\Psi'_{2n} = \lim_{m\to\infty} \Psi'_{2n,m}$ and $\Phi'_{2n} = \lim_{m\to\infty} \Phi'_{2n,m}$. These sets are finite; we fix an m large enough that $\Psi'_{2n} \simeq \Psi'_{2n,m}$ and $\Phi'_{2n} \simeq \Phi'_{2n,m}$. Elements of Φ'_{2n} (resp. Ψ'_{2n}) are called *symbols* (resp. *u-symbols*).

The bijection $i : \Psi'_{2n} \to \Phi'_{2n}$ is given by

$$i(a_0, \ldots, a_{2m}) = (a_0, a_1 - 1, a_2 - 1, \ldots, a_{2m-1} - m, a_{2m} - m).$$

Definition 4.1. A *block* (for the distinguished u-symbol $\mathbf{a}$) is a subset $[k, l]$ of $[0, 2m]$ that satisfies one of the following conditions:

- $[k, l]$ is a union of consecutive parts $P_1, \ldots, P_r$, where P_1 and P_r are ladders, P_r is the unique part with even top, and P_1 is the unique part with odd bottom.
- $k = 0$, and $[0, l]$ is the union of consecutive parts $P_2, \ldots, P_r$, where P_r is a ladder and the only part with even top and where all parts have even bottom.

In either case, the *top* of the block is the top of P_r, and its *bottom* is the bottom of P_1, provided the latter is defined. Blocks satisfying the second condition above are said to have no bottom. Note that there is always exactly one block of this type.

We remark that it is permitted that $r = 1$, in which case the block consists of a single ladder. A part belongs to some block if and only if it is not a staircase with odd bottom (and hence even top).

Lemma 4.2. *For any $\mathbf{a} \in \Psi'_{2n}$, there are an odd number of isolated points. Suppose $i(\mathbf{a}) = (a'_0, \dots, a'_{2m})$, and let $k_0, \dots, k_{2f}$ be the isolated points, numbered such that*

$$a'_{k_0} < a'_{k_1} < \cdots < a'_{k_{2f}}.$$

If $i(\mathbf{a})$ is special, then $k_t \equiv t \pmod 2$.

Proof. It is obvious from the definition of symbol that the even and odd indexed entries are strictly increasing, so that an entry can appear at most twice. Hence, the number of isolated points is odd; it is the length of the symbol minus twice the number of duplicated entries.

Now suppose that $i(\mathbf{a})$ is special. Every entry before a'_{k_0} appears twice, so k_0 is even. Similarly, every entry between a'_{k_t} and $a'_{k_{t+1}}$ is repeated, so $k_{t+1} \equiv k_t + 1 \equiv t + 1 \pmod 2$ by induction. □

We will adhere to the convention introduced in this lemma for naming the isolated points. In particular, this means that if $i(\mathbf{a})$ is not special, it is not necessarily the case that $k_t < k_{t+1}$.

If $i(\mathbf{a})$ is a special symbol, we identify $V(i(\mathbf{a}))$ with the space $V_{\mathbf{c}}$ of Section 3.1 (where $\mathbf{c}$ is the two-sided cell corresponding to $i(\mathbf{a})$) by

$$e_i = \{k_{i-1}, k_i\}, \qquad i = 1, \dots, 2f; \qquad\qquad e_0 = \{k_1, k_2, \dots, k_{2f}\}.$$

Lemma 4.3. *Let $\mathbf{a} = (a_0, \dots, a_{2m})$ be a distinguished u-symbol, and let $\mathbf{a}' = i(\mathbf{a}) = (a'_0, \dots, a'_{2m})$ be the corresponding symbol. Let k and l be two integers such that $0 \le k < l \le 2m$.*

1. *If k and l belong to distinct parts, then $a'_k < a'_l$.*
2. *If $[k, l]$ is a staircase with even bottom, then $k+1$ and $l-1$ are the only isolated points in $[k, l]$. A staircase with odd bottom contains no isolated points.*
3. *If $[k, l]$ is a ladder, then k is an isolated point if it is odd, and l is an isolated point if it is even. There are no other isolated points in $[k, l]$.*
4. *The number of isolated points in any part is congruent to the length of the part modulo 2.*
5. *For each t, either $k_t < k_{t+1}$, or $k_t = k_{t+1} + 1$ and $[k_{t+1}, k_t]$ is a staircase with even bottom.*
6. *Displaced isolated points occur in pairs $\{k_t, k_{t+1}\} = e_{t+1}$ with t even, with one such pair for each staircase with even bottom.*

Proof. (1) If $k \equiv l \pmod 2$ but $k \ne l$, it follows from the definition of Φ'_{2n} that $a'_k < a'_l$. Indeed, the fact that $a'_i \le a'_{i+2} - 1$ implies that $a'_i \le a'_{i+2j} - j$. We need only treat the case where $k \not\equiv l \pmod 2$.

Assume now that $k < l$, k is even, and l is odd. Since $a_{l-1} \le a_l$, $a'_{l-1} = a_{l-1} - (l-1)/2$, and $a'_l = a_l - (l+1)/2$, it follows that $a'_l \ge a'_{l-1} - 1$. Therefore,

assuming $k < l-3$, we have

$$a'_k \leq a'_{l-1} - (l-1-k)/2 < a'_{l-1} - 1 \leq a'_l.$$

If $k = l-1$, the fact that k and l are in different parts implies that $a_{l-1} \leq a_l - 2$, whence $a'_l \geq a'_{l-1} + 1$. Thus, $a'_k < a'_l$.

Finally, suppose $k = l-3$. If $l-1$ and l are in different parts, then the previous case gives $a'_l \geq a'_{l-1} + 1 \geq a'_{l-3} + 2$. Similar reasoning applies if $l-3$ and $l-2$ are in different parts. Otherwise, we have $l-3$ and $l-2$ in one part and $l-1$ and l in another. This means that $(a_{l-3}, \ldots, a_l)$ has the form (a, a, b, b), $(a-1, a, b, b)$, or $(a, a, b, b+1)$ with $b - a > 2$ or $(a-1, a, b, b+1)$ with $b - a > 1$. In each case, $a_l > a_{l-3} + 2$, so that $a'_l = a_l - (l+1)/2 > a_{l-3} - (l-3)/2 = a'_{l-3}$.

The reasoning is similar if k is odd and l is even.

(2) Suppose $(a_k, a_{k+1}, \ldots, a_l) = (a, a, a+2, a+2, \ldots, a+l-k-1)$. For $i \in [k, l]$, it is easy to verify that:

$$a'_i = \begin{cases} a - k + i/2 & \text{if } k \text{ is even and } i \text{ is even,} \\ a - k + (i-3)/2 & \text{if } k \text{ is even and } i \text{ is odd,} \\ a - k + (i-2)/2 & \text{if } k \text{ is odd and } i \text{ is even,} \\ a - k + (i-1)/2 & \text{if } k \text{ is odd and } i \text{ is odd.} \end{cases}$$

If k is odd, then $a'_i = a'_{i+1}$ for $i = k, k+2, \ldots, l-1$, so there are no isolated points. On the other hand, if k is even, then $a'_i = a'_{i+3}$ for $i = k, k+2, \ldots, l-3$, so none of the integers

$$k, k+2, \ldots, l-3; \quad k+3, k+5, \ldots, l$$

are isolated. However, a'_{k+1} and a'_{l-1} are not duplicated. In view of part (1), we see that $k+1$ and $l-1$ are isolated.

(3) Suppose $(a_k, a_{k+1}, \ldots, a_l) = (a, a+1, \ldots, a+l-k)$. For $i \in [k, l]$, we have

$$a'_i = \begin{cases} a - k + i/2 & \text{if } i \text{ is even,} \\ a - k + (i-1)/2 & \text{if } i \text{ is odd.} \end{cases}$$

Thus, $a'_i = a'_{i+1}$ if i is even and $i, i+1 \in [k, l]$. It follows that k is isolated if and only if it is odd, that l is isolated if and only if it is even, and that no other points can be isolated.

(4) From the preceding parts of the lemma, we see that each staircase and each ladder of even length contains 0 or 2 isolated points, whereas each ladder of odd length contains exactly one isolated point.

(5) From part (1) we see that the inequality $k_t < k_{t+1}$ can be violated only if k_t and k_{t+1} belong to the same part. Examining the formulas above for isolated points in ladders and staircases yields the result.

(6) From parts (3) and (4), it follows that no isolated points in ladders can be displaced, while parts (2) and (4) together imply that all isolated points in staircases are displaced. Finally, if $[k, l]$ is a staircase with even bottom, then $k_t = k+1$, so that k_t is odd. Since a special symbol has $k_t \equiv t \pmod 2$, we see that t is even. □

The following proposition is now an immediate consequence of the definition of blocks and parts (2) and (3) of the preceding lemma.

Proposition 4.4. *Let* $\mathbf{a}$ *be a distinguished* u*-symbol. The top and bottom of each block of* $\mathbf{a}$ *are nondisplaced isolated points. In addition to these, each block contains an even number (possibly zero) of additional isolated points, all of which are displaced. There are no isolated points that do not belong to any block.* □

Proposition 4.5. *For any block* b *of a distinguished* u*-symbol* $\mathbf{a}$*, the element* $m_b \in \mathcal{G}_{\mathbf{c}} \times \hat{\mathcal{G}}_{\mathbf{c}}$ *is of the form* $(x, 1)$ *for some* $x \in \mathcal{G}_{\mathbf{c}}$*.*

Proof. It follows immediately from Lemma 4.3(6) that $\tilde{\kappa}(\mathbf{a})$ is a sum of e_i's with i odd. Thus, $\tilde{\kappa}(\mathbf{a}) \in \mathcal{G}_{\mathbf{c}} \times \{1\} \subset M(\mathcal{G}_{\mathbf{c}})$. □

If C is the unipotent class corresponding to $\mathbf{a}$, then we set $t(C)$ equal to the x appearing in the above proposition.

Proposition 4.6. *If* C_1 *and* C_2 *are two unipotent classes in the same special piece, then* $t(C_1) \leq t(C_2)$ *in the partial order on* $\mathcal{G}_{\mathbf{c}}$ *if and only if* C_1 *is in the closure of* C_2*.*

Proof. Let $\mathbf{a}$ be a distinguished u-symbol corresponding to a unipotent class C, and suppose that k_{t-1} and k_t are consecutive nondisplaced isolated points of $\mathbf{a}$ with t odd. Thus, if we write $\tilde{\kappa}(\mathbf{a}) = (x, 1)$, then the basis element $e_t \in \mathcal{G}_{\mathbf{c}}$ does not occur in x. Now, assume that the twisted symbol $i(\mathbf{a})^{e_t}$ is defined, and let $\mathbf{a}'$ be the u-symbol satisfying $i(\mathbf{a}') = i(\mathbf{a})^{e_t}$. Furthermore, let us assume that $\mathbf{a}'$ is also distinguished, corresponding to the unipotent class C'. Evidently, we have $\tilde{\kappa}(\mathbf{a}') = (x + e_t, 1)$ and $x + e_t < x$. The first step will be to prove that in this context, C' lies in the closure of C. We introduce the following notation:

$$\begin{aligned} \mathbf{a} &= (a_0, \ldots, a_{2m}) & i(\mathbf{a}) &= \mathbf{b} = (b_0, \ldots, b_{2m}) \\ \mathbf{a}' &= (a'_0, \ldots, a'_{2m}) & i(\mathbf{a}') &= \mathbf{b}^{e_t} = \mathbf{b}' = (b'_0, \ldots, b'_{2m}) \end{aligned}$$

Now, from the description of isolated points in Lemma 4.3 and Proposition 4.4, we know that k_{t-1} must be the top of a block and k_t the bottom of the next block. Therefore, $[k_{t-1} + 1, k_t - 1]$ is a (possibly empty) union of staircases with odd bottom. In particular, $a_i = a_{i+1}$ for all odd $i \in [k_{t-1} + 1, k_t - 1]$, and the formula for the bijection i gives $b_i = b_{i+1}$ for all odd $i \in [k_{t-1} + 1, k_t - 1]$ as well. (On the other hand, we note for later reference that $a_i \leq a_{i+1} - 2$ for all even $i \in [k_{t-1}, k_t]$.) Using these observations, one can show that the twisted symbol is described by

$$b'_i = \begin{cases} b_i & \text{if } i \notin [k_{t-1}, k_t], \\ b_{i+1} & \text{if } i \in [k_{t-1}, k_t] \text{ and } i \text{ is even}, \\ b_{i-1} & \text{if } i \in [k_{t-1}, k_t] \text{ and } i \text{ is odd}. \end{cases}$$

(It suffices to verify that the sequence $(b_0, \ldots, b_{2m})$ is a valid symbol and that it satisfies the parity conditions in the definition of twisting.)

Again using the formula for i, we find that

$$a'_i = \begin{cases} a_i & \text{if } i \notin [k_{t-1}, k_t], \\ a_{i+1} - 1 & \text{if } i \in [k_{t-1}, k_t] \text{ and } i \text{ is even}, \\ a_{i-1} + 1 & \text{if } i \in [k_{t-1}, k_t] \text{ and } i \text{ is odd}. \end{cases}$$

Since $a_i \leq a_{i+1} - 2$ for all even $i \in [k_{t-1}, k_t]$, we observe in particular that if $i \in [k_{t-1}, k_t]$ is even, then

$$a'_i > a_i, \qquad a'_{i+1} < a_{i+1}, \qquad \text{and} \quad a'_i + a'_{i+1} = a_i + a_{i+1}.$$

Now, for any u-symbol $\mathbf{c} = (c_0, \ldots, c_{2m})$, we define the sums

$$\sigma_j(\mathbf{c}) = \sum_{i=j}^{2m} c_i.$$

From the above formulas, it is easy to see that

$$\begin{aligned} \sigma_j(\mathbf{a}') < \sigma_j(\mathbf{a}) &\quad \text{if } j \in [k_{t-1}, k_t] \text{ is odd, and} \\ \sigma_j(\mathbf{a}') = \sigma_j(\mathbf{a}) &\quad \text{for all other } j \in [0, 2m]. \end{aligned}$$

It was shown in [2, Lemme 3.2] that C' is contained in the closure of C if and only if $\sigma_j(\mathbf{a}') \leq \sigma_j(\mathbf{a})$ for all j. So C' is indeed contained in the closure of C.

More generally, if C_1 and C_2 are two unipotent classes in the same special piece with the property that $t(C_1) < t(C_2)$, then $t(C_1)$ contains various terms e_i (with i odd) not appearing in $t(C_2)$. Adding these terms to $t(C_2)$ one at a time and iterating the above argument shows that C_1 is contained in the closure of C_2.

It remains to prove the opposite implication. We will show that if $t(C_1)$ and $t(C_2)$ are incomparable, then so are C_1 and C_2. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be the corresponding distinguished u-symbols, and let $\mathbf{a}_0$ be the u-symbol corresponding to the special class in their special piece. Let $k_0, \ldots, k_{2f}$ be the isolated points of $\mathbf{a}_0$. Since $t(C_1)$ is not contained in $t(C_2)$, there exists a pair of consecutive isolated points $\{k_{s-1}, k_s\}$ with s odd, such that the corresponding isolated points of $\mathbf{a}_1$ are displaced, but those of $\mathbf{a}_2$ are not. In other words, the term e_s appears in $t(C_1)$, but not in $t(C_2)$. On the other hand, since $t(C_2) \nsubseteq t(C_1)$, there exists another such pair $\{k_{t-1}, k_t\}$ with t odd, which is displaced in $\mathbf{a}_2$, but not in $\mathbf{a}_1$. Choose an odd number j_1 between k_{s-1} and k_s and another odd number j_2 between k_{t-1} and k_t. The above calculations of σ_j show that

$$\begin{aligned} \sigma_{j_1}(\mathbf{a}_1) <& \sigma_{j_1}(\mathbf{a}_0) = \sigma_{j_1}(\mathbf{a}_2), \\ \sigma_{j_2}(\mathbf{a}_1) =& \sigma_{j_2}(\mathbf{a}_0) > \sigma_{j_2}(\mathbf{a}_2). \end{aligned}$$

Again invoking [2, Lemme 3.2], we see that this pair of inequalities implies that neither C_1 nor C_2 may be contained in the closure of the other. $\square$

Proposition 4.7. *Let* $\mathbf{a}$ *be a distinguished* u*-symbol, corresponding to the unipotent class* C*. The elements* $\pi(m_b)$*, as* b *runs over the blocks of* $\mathbf{a}$*, are the components of* $t(C)$*, and the elements* $\pi(\lambda_b)$ *are the images in* $V_{\mathbf{c}}$ *of the blocks of* $t(C)$*.*

Proof. Let b a block of $\mathbf{a}$. First, assume that b has a bottom. Let $k_{t-1}, k_t, \ldots, k_{t+2j}$ be the isolated points of b, where k_{t-1} is the bottom of b and k_{t+2j} is its top. Since k_{t-1} is odd and nondisplaced, t is even. Moreover, k_{t-2} cannot be displaced, because it must be the top of the preceding block, while k_{t+2j+1} is either undefined or the bottom of the next block. This implies that the pairs $\{k_{t-2}, k_{t-1}\}$ and $\{k_{t+2j}, k_{t+2j+1}\}$ contain no displaced isolated points, so the terms e_{t-1} and e_{t+2j+1} (provided the latter is defined) do not appear in $\tilde{\kappa}(\mathbf{a})$. Accordingly, $e_t + e_{t+2} + \cdots + e_{t+2j}$ is a block of $t(C)$ with associated component $e_{t+1} + e_{t+3} + \cdots + e_{t+2j-1}$. Projecting to $V_{\mathbf{c}}$, we obtain

$$\begin{aligned} \pi(\lambda_b) &= \pi(\{k_{t-1}, \ldots, k_{t+2j}\}) = e_t + e_{t+2} + \cdots + e_{t+2j}, \\ \pi(m_b) &= \pi(\{k_t, \ldots, k_{t+2j-1}\}) = e_{t+1} + e_{t+3} + \cdots + e_{t+2j-1}, \end{aligned}$$

where we abuse notation slightly by viewing the e_i's as elements of $V_{\mathbf{c}}$.

If b has no bottom, then the isolated points of b are of the form $k_0, k_1, \dots, k_{2j}$. Just as before, $e_0 + e_2 + \cdots + e_{2j}$ is a block of $t(C)$ with associated component $e_1 + e_3 + \cdots + e_{2j-1}$ while the above formula for $\pi(m_b)$ is still valid, with $t = 0$. Recall,however, that the definition of λ_b is different when b contains an odd number of isolated points: if $\mathbf{a}$ contains $2f + 1$ isolated points in all, then

$$\pi(\lambda_b) = \pi(\{k_{2j+1}, k_{2j+2}, \dots, k_{2f}\}) = e_{2j+2} + e_{2j+4} + \cdots + e_{2f}.$$

Nevertheless, this is still the image in $V_{\mathbf{c}}$ of a block, namely $e_0 + e_2 + \cdots + e_{2j}$. $\square$

Proposition 4.8. *The set $\mathcal{T}(\mathbf{a})$ of elements $b \in \bar{H}(\mathbf{a})$ for which $\mathbf{a}^b$ is defined is the codimension one hyperplane consisting of all subsets not containing the unique bottomless block. Moreover, for all such b, $i(\mathbf{a}^b) = (i(\mathbf{a}))^{\lambda_b}$.*

Proof. It suffices to prove that $\mathbf{a}^b$ is defined and that the formula for $i(\mathbf{a}^b)$ holds for the case where $b = [k, l]$ is a single block with a bottom. We introduce the following notation:

$$\begin{aligned} \mathbf{a} &= (a_0, \dots, a_{2m}) & i(\mathbf{a}) &= (b_0, \dots, b_{2m}) \\ \mathbf{a}^b &= (a'_0, \dots, a'_{2m}) & i(\mathbf{a})^{\lambda_b} &= (b'_0, \dots, b'_{2m}) \\ i(\mathbf{a}^b) &= (c_0, \dots, c_{2m}) \end{aligned}$$

Now, from the definition of block, it follows that $a_{k-1} + 1 < a_i < a_{l+1} - 1$ for all $i \in [k, l]$ (as long as a_{l+1} is defined). From this observation, one can deduce that

$$a'_i = \begin{cases} a_i & \text{if } i \notin [k, l], \\ a_{i-1} & \text{if } i \in [k, l] \text{ and } i \text{ is even}, \\ a_{i+1} & \text{if } i \in [k, l] \text{ and } i \text{ is odd}. \end{cases}$$

Indeed, it suffices to check that the sequence defined by this formula is a valid u-symbol and that it satisfies the parity conditions in the definition of twisting. Both of these are easy to verify.

Next, it follows from the calculations in the proofs of parts (2) and (3) of Lemma 4.3 that

$$\{b_i \mid i \in [k, l] \text{ is odd and not isolated}\} = \{b_i \mid i \in [k, l] \text{ is even and not isolated}\}.$$

This observation lets us conclude that $i(\mathbf{a})^{\lambda_b}$ is defined and

$$b'_i = \begin{cases} b_i & \text{if } i \notin [k, l], \\ b_{i-1} & \text{if } i \in [k, l] \text{ and } i \text{ is even}, \\ b_{i+1} & \text{if } i \in [k, l] \text{ and } i \text{ is odd}. \end{cases}$$

Finally, from the formula for i, we have

$$b_i = \begin{cases} a_i - i/2 \\ a_i - (i+1)/2 \end{cases} \qquad \text{and} \qquad c_i = \begin{cases} a'_i - i/2 & \text{if } i \text{ is even}, \\ a'_i - (i+1)/2 & \text{if } i \text{ is odd}. \end{cases}$$

It follows that

$$c_i = \begin{cases} a_i - i/2 & \text{if } i \notin [k,l] \text{ and } i \text{ is even,} \\ a_i - (i+1)/2 & \text{if } i \notin [k,l] \text{ and } i \text{ is odd,} \\ a_{i-1} - i/2 & \text{if } i \in [k,l] \text{ and } i \text{ is even,} \\ a_{i+1} - (i+1)/2 & \text{if } i \in [k,l] \text{ and } i \text{ is odd} \end{cases}$$

$$= \begin{cases} b_i & \text{if } i \notin [k,l], \\ b_{i-1} & \text{if } i \in [k,l] \text{ and } i \text{ is even,} \\ b_{i+1} & \text{if } i \in [k,l] \text{ and } i \text{ is odd} \end{cases} = b'_i.$$

This is the desired equality. □

Proposition 4.9. *The image of* $p|_{\bar{H}(\mathbf{a})} : \bar{H}(\mathbf{a}) \to \mathrm{Irr}(A(C))$ *is precisely* $\mathrm{Irr}(\bar{A}(C))$.

Proof. Recall that $\bar{A}(C)$ is defined to be the smallest quotient of $A(C)$ such that all local systems on C that are assigned by the Springer correspondence to representations in the same two-sided cell as $\chi_{C,1}$ actually come from representations of $\bar{A}(C)$. Equivalently, $\mathrm{Irr}(\bar{A}(C))$ can be characterized as the smallest subgroup of $\mathrm{Irr}(A(C))$ that enjoys the following property:

> For any u-symbol $\mathbf{a}'$ that is similar to $\mathbf{a}$, $\nu(\mathbf{a}') \in \mathrm{Irr}(\bar{A}(C))$ if $i(\mathbf{a}')$ is congruent to $i(\mathbf{a})$.

We first show that the image of $p|_{\bar{H}(\mathbf{a})}$ lies in $\mathrm{Irr}(A(C))$. Since $\mathcal{T}(\mathbf{a})$ is a transversal of the kernel, it suffices to show that $p(\mu) \in \mathrm{Irr}(A(C))$ for each $\mu \in \mathcal{T}(\mathbf{a})$. For such a μ, the u-symbol $\mathbf{a}^\mu$ is defined and is similar to $\mathbf{a}$. Moreover, the preceding proposition tells us that $i(\mathbf{a}^\mu) = i(\mathbf{a})^{\lambda_\mu}$. Thus, $i(\mathbf{a}^\mu)$ and $i(\mathbf{a})$ are congruent, so $p(\mu) = \nu(\mathbf{a}^\mu)$ lies in $\mathrm{Irr}(\bar{A}(C))$.

We use a dimension argument to prove that the image of $p|_{\bar{H}(\mathbf{a})}$ is all of $\mathrm{Irr}(\bar{A}(C))$. It is a consequence of Proposition 3.13 that the image is an $\mathbb{F}_2$-vector space whose dimension is given by any of the following:

$$\begin{aligned} (\text{number of blocks in } \mathbf{a}) - 1 \quad &= \quad (\text{number of blocks with a bottom}) \\ &= \quad (\text{number of ladders with odd bottom}) \end{aligned}$$

We now show that this is also the dimension of $\bar{A}(C)$. Recall that unipotent classes in type C are in one-to-one correspondence with partitions of $2n$ in which odd parts occur with even multiplicity. Let λ be the partition corresponding to the same unipotent class to which $\mathbf{a}$ corresponds. Write λ as

$$\lambda = (0 \le \lambda_0 \le \cdots \le \lambda_{2m})$$

by increasing m or adding zero parts as necessary. In the course of the proof of [2, Lemme 3.2], a direct formula for $\mathbf{a}$ in terms of λ is obtained:

$$a_i = \begin{cases} \frac{1}{2}\lambda_i + i + 1 & \text{if } \lambda_i \text{ is even,} \\ \frac{1}{2}\lambda_i + i + \frac{1}{2} & \text{if } \lambda_i \text{ is odd and } |\{j > i \mid \lambda_j \text{ odd}\}| \text{ is even,} \\ \frac{1}{2}\lambda_i + i + \frac{3}{2} & \text{if } \lambda_i \text{ is odd and } |\{j > i \mid \lambda_j \text{ odd}\}| \text{ is odd.} \end{cases} \tag{6}$$

(In [2], the entries of λ and of $\mathbf{a}$ (there called μ) are indexed by i with $1 \le i \le 2m+1$. Since we index them here by i with $0 \le i \le 2m$, we must add 1 to the formulas given there for μ_i.) Now, consider a fragment of the partition:

$$\cdots \le \lambda_{k-1} < \lambda_k = \cdots = \lambda_l < \lambda_{l+1} \le \cdots.$$

Let p be the common value of $\lambda_k, \dots, \lambda_l$. If p is even, the above formula for $\mathbf{a}$ makes it clear that $[k, l]$ is a ladder. On the other hand, if p is odd, the fact that every odd part has even multiplicity means that a_k must be calculated using the third case in (6), a_{k+1} must be calculated using the second case, and so on. From this, it is easy to see that $[k, l]$ must be a staircase. Now, define the height of the part p to be

$$|\{j \mid \lambda_j < p\}| + 1.$$

(Here, we add 1 to make our usage of "height" consistent with [10]; partitions in type C are there assumed to have an even number of parts, whereas we assume that they have an odd number of parts.) Evidently, the height of p is simply $k + 1$.

In particular, even parts of even height in λ correspond to ladders with odd bottom in $\mathbf{a}$. According to the last paragraph of [10, Section 5], the dimension of $\bar{A}(C)$ is precisely the number of even parts of even height in λ. Thus, the image of $p|_{\bar{H}(\mathbf{a})}$ must be the full character group $\mathrm{Irr}(\bar{A}(C))$. □

4.2. **Type** B**.** We let $\Psi_{2n+1,m}$ (resp. $\Phi_{2n+1,m}$) denote the set of sequences $\mathbf{a} = (a_0, a_1, \dots, a_{2m})$ satisfying the following conditions:

$$\begin{aligned} &0 \le a_0, \quad 0 \le a_1, \quad a_i \le a_{i+2} - 2, \quad \sum a_i = n + 2m^2 \quad \text{for } \mathbf{a} \in \Psi_{2n+1,m}, \\ &0 \le a_0, \quad 0 \le a_1, \quad a_i \le a_{i+2} - 1, \quad \sum a_i = n + m^2 \quad \text{for } \mathbf{a} \in \Phi_{2n+1,m}. \end{aligned}$$

The shift operations $S : \Psi_{2n+1,m} \to \Psi_{2n+1,m+1}$, $S : \Phi_{2n+1,m} \to \Phi_{2n+1,m+1}$ are given by

$$\begin{aligned} S(\mathbf{a}) &= (0, 0, a_0 + 2, \dots, a_{2m} + 2) \qquad \text{for } \mathbf{a} \in \Psi_{2n+1,m}, \\ S(\mathbf{a}) &= (0, 0, a_0 + 1, \dots, a_{2m} + 1) \qquad \text{for } \mathbf{a} \in \Phi_{2n+1,m}. \end{aligned}$$

As before, we put $\Psi_{2n+1} = \lim_{m\to\infty} \Psi_{2n+1,m}$ and $\Phi_{2n+1} = \lim_{m\to\infty} \Phi_{2n+1,m}$. These sets are again finite; we fix an m large enough that $\Psi_{2n+1} \simeq \Psi_{2n+1,m}$ and $\Phi_{2n+1} \simeq \Phi_{2n+1,m}$. Elements of Φ_{2n+1} (resp. Ψ_{2n+1}) are called *symbols* (resp. *u-symbols*).

The bijection $i : \Psi_{2n+1} \to \Phi_{2n+1}$ is given by

$$i(a_0, \dots, a_{2m}) = (a_0, a_1, a_2 - 1, a_3 - 1, \dots, a_{2m} - m).$$

Definition 4.10. A *block* (for the distinguished u-symbol $\mathbf{a}$) is a subset $[k, l]$ of $[0, 2m]$ that satisfies one of the following conditions:

- $[k, l]$ is a union of consecutive parts $P_1, \dots, P_r$, where P_1 and P_r are ladders, P_1 is the unique part with even bottom, and P_r is the unique part with odd top.
- $l = 2m$, and $[k, 2m]$ is the union of consecutive parts $P_1, \dots, P_{r-1}$, where P_1 is a ladder and the only part with even bottom and where all parts have even top.

In either case, the *bottom* of the block is the bottom of P_1, and its *top* is the top of P_r, provided the latter is defined. Blocks satisfying the second condition above are said to have no top. There is always exactly one block of this type.

Note that a part belongs to some block if and only if it is not a staircase with even bottom (and hence odd top).

Lemma 4.11. *The set* $\mathcal{T}(\mathbf{a})$ *of elements* $b \in \bar{H}(\mathbf{a})$ *for which* $\mathbf{a}^b$ *is defined is the codimension one hyperplane consisting of all subsets not containing the unique topless block.*

Lemma 4.12. *For any* $\mathbf{a} \in \Psi_{2n+1}$*, there are an odd number of isolated points. Suppose* $i(\mathbf{a}) = (a'_0, \ldots, a'_{2m})$*, and let* $k_0, \ldots, k_{2f}$ *be the isolated points, numbered such that*

$$a'_{k_0} < a'_{k_1} < \cdots < a'_{k_{2f}}.$$

If $i(\mathbf{a})$ *is special, then* $k_t \equiv t \pmod 2$*.*

As we did in type C, we will adhere to the convention introduced in this lemma for naming the isolated points. If $i(\mathbf{a})$ is a special symbol, we identify $V(i(\mathbf{a}))$ with the space $V_{\mathbf{c}}$ of Section 3.1 (where $\mathbf{c}$ is the two-sided cell corresponding to $i(\mathbf{a})$) by

$$e_i = \{k_i, k_{i+1}\}, \qquad i = 0, \ldots, 2f-1; \qquad\qquad e_{2f} = \{k_0, k_1, \ldots, k_{2f-1}\}.$$

Lemma 4.13. *Let* $\mathbf{a} = (a_0, \ldots, a_{2m})$ *be a distinguished* u*-symbol, and let* $\mathbf{a}' = i(\mathbf{a}) = (a'_0, \ldots, a'_{2m})$ *be the corresponding symbol. Let* k *and* l *be two integers such that* $0 \le k < l \le 2m$*.*

(1) *If* k *and* l *belong to distinct parts, then* $a'_k < a'_l$*.*
(2) *If* $[k, l]$ *is a staircase with odd bottom, then* $k+1$ *and* $l-1$ *are the only isolated points in* $[k, l]$*. A staircase with even bottom contains no isolated points.*
(3) *If* $[k, l]$ *is a ladder, then* k *is an isolated point if it is even, and* l *is an isolated point if it is odd. There are no other isolated points in* $[k, l]$*.*
(4) *The number of isolated points in any part is congruent to the length of the part modulo* 2*.*
(5) *For each* t*, either* $k_t < k_{t+1}$*, or* $k_t = k_{t+1} + 1$ *and* $[k_{t+1}, k_t]$ *is a staircase with odd bottom.*
(6) *Displaced isolated points occur in pairs* $\{k_t, k_{t+1}\} = e_t$ *with* t *odd, with one such pair for each staircase with odd bottom.*

4.3. **Type** D**.** Finally, let $\Psi_{2n,m}$ (resp. $\Phi_{2n,m}$) denote the set of sequences $\mathbf{a} = (a_0, a_1, \ldots, a_{2m+1})$ satisfying the following conditions:

$$0 \le a_0, \quad 0 \le a_1, \quad a_i \le a_{i+2} - 2, \quad \sum a_i = n + 2m^2 + 2m \quad \text{for } \mathbf{a} \in \Psi_{2n,m},$$

$$0 \le a_0, \quad 0 \le a_1, \quad a_i \le a_{i+2} - 1, \quad \sum a_i = n + m^2 + m \quad \text{for } \mathbf{a} \in \Phi_{2n,m},$$

together with the additional condition:

($\star$) if i is the smallest index such that $a_{2i} \ne a_{2i+1}$, then $a_{2i} < a_{2i+1}$.

(Note that such an i need not exist; in the case of u-symbols, there is no such i if and only if $\mathbf{a}$ is a union of staircases, necessarily all with even bottom.)

The shift operations $S : \Psi_{2n,m} \to \Psi_{2n,m+1}$, $S : \Phi_{2n,m} \to \Phi_{2n,m+1}$ are given by

$$S(\mathbf{a}) = (0, 0, a_0 + 2, \ldots, a_{2m+1} + 2) \qquad \text{for } \mathbf{a} \in \Psi_{2n,m},$$

$$S(\mathbf{a}) = (0, 0, a_0 + 1, \ldots, a_{2m+1} + 1) \qquad \text{for } \mathbf{a} \in \Phi_{2n,m}.$$

Put $\Psi_{2n} = \lim_{m\to\infty} \Psi_{2n,m}$ and $\Phi_{2n} = \lim_{m\to\infty} \Phi_{2n,m}$. We fix an m large enough that $\Psi_{2n} \simeq \Psi_{2n,m}$ and $\Phi_{2n} \simeq \Phi_{2n,m}$. Elements of Φ_{2n} (resp. Ψ_{2n}) are called *symbols* (resp. *u-symbols*).

The bijection $i : \Psi_{2n} \to \Phi_{2n}$ is given by

$$i(a_0, \dots, a_{2m+1}) = (a_0, a_1, a_2 - 1, a_3 - 1, \dots, a_{2m} - m, a_{2m+1} - m).$$

Definition 4.14. A subset $[k,l]$ of $[0,2m]$ is called a *block* (for the distinguished u-symbol $\mathbf{a}$) if it is a union of consecutive parts $P_1, \dots, P_r$, where P_1 and P_r are ladders, P_r is the unique part with odd top, and P_1 is the unique part with even bottom. The *top* of the block is the top of P_r, and its *bottom* is the bottom of P_1.

A part belongs to some block if and only if it is not a staircase with even bottom (and hence odd top).

In type D, it is possible for a u-symbol $\mathbf{a}$ to have no blocks and no isolated points; this occurs precisely when $\mathbf{a}$ is a union of staircases. In this situation, $\bar{A}(C)$ and $\mathcal{G}'_{\mathbf{c}}$ both have only one element, and the main theorem is trivial. For the rest of this section, we assume where necessary that $\mathbf{a}$ has a block (and accordingly, an isolated point).

The sequence of integers $\mathbf{a}^\mu$ is defined for each $\mu \in \bar{H}(\mathbf{a})$, but it is not necessarily a u-symbol as it need not satisfy condition $(\star)$.

Lemma 4.15. *For each $\mu \in \bar{H}(\mathbf{a})$, exactly one of $\mathbf{a}^\mu$ and $\mathbf{a}^{\mu^c}$ satisfies condition $(\star)$. Thus, $\mathcal{T}(\mathbf{a})$ is a transversal of the subgroup $\{\varnothing, B(\mathbf{a})\}$.*

Lemma 4.16. *For any $\mathbf{a} \in \Psi_{2n}$, there are an even number of isolated points. Suppose $i(\mathbf{a}) = (a'_0, \dots, a'_{2m+1})$, and let $k_0, \dots, k_{2f+1}$ be the isolated points, numbered such that*

$$a'_{k_0} < a'_{k_1} < \cdots < a'_{k_{2f+1}}.$$

If $i(\mathbf{a})$ is special, then $k_t \equiv t \pmod 2$.

If $i(\mathbf{a})$ is a special symbol, we identify $V(i(\mathbf{a}))$ with the space $\tilde{V}_{\mathbf{c}}$ of Section 3.1 (where $\mathbf{c}$ is the two-sided cell corresponding to $i(\mathbf{a})$) by

$$e_i = \{k_i, k_{i+1}\}, \qquad i = 0, \dots, 2f.$$

Note that in type D, the map $\pi : V(i(\mathbf{a})) \to V_{\mathbf{c}}$ is not an isomorphism.

Lemma 4.17. *Let $\mathbf{a} = (a_0, \dots, a_{2m+1})$ be a distinguished u-symbol, and let $\mathbf{a}' = i(\mathbf{a}) = (a'_0, \dots, a'_{2m+1})$ be the corresponding symbol. Let k and l be two integers such that $0 \le k < l \le 2m+1$.*

1. *If k and l belong to distinct parts, then $a'_k < a'_l$.*
2. *If $[k,l]$ is a staircase with odd bottom, then $k+1$ and $l-1$ are the only isolated points in $[k,l]$. A staircase with even bottom contains no isolated points.*
3. *If $[k,l]$ is a ladder, then k is an isolated point if it is even, and l is an isolated point if it is odd. There are no other isolated points in $[k,l]$.*
4. *The number of isolated points in any part is congruent to the length of the part modulo 2.*
5. *For each t, either $k_t < k_{t+1}$, or $k_t = k_{t+1} + 1$ and $[k_{t+1}, k_t]$ is a staircase with odd bottom.*
6. *Displaced isolated points occur in pairs $\{k_t, k_{t+1}\} = e_t$ with t odd, with one such pair for each staircase with odd bottom.*

References

[1] P. Achar, *An order-reversing duality map for conjugacy classes in Lusztig's canonical quotient*, Transform. Groups **8** (2003), 107–145.

[2] P. Achar and A.-M. Aubert, *Supports unipotents de faisceaux caractères*, to appear in J. Inst. Math. Jussieu.

[3] R. W. Carter, *Finite groups of Lie type: conjugacy classes and complex characters*, Pure and Applied Mathematics, John Wiley & Sons, Inc., New York, 1985.

[4] H. Kraft and C. Procesi, *A special decomposition of the unipotent variety*, in *Orbites unipotentes et représentations, III*, Astérisque No. 173–174 (1989), 271–279.

[5] G. Lusztig, *A class of irreducible representations of a Weyl group*, Nederl. Akad. Wetensch. Indag. Math. **41** (1979), 323–335.

[6] G. Lusztig, *Characters of reductive groups over a finite field*, Ann. Math. Studies no. 107, Princeton University Press, 1984.

[7] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), 205–272.

[8] G. Lusztig, *Character sheaves, IV*, Adv. Math. **59** (1986), 1–63.

[9] G. Lusztig, *Notes on unipotent classes*, Asian J. Math. **1** (1997), 194–207.

[10] E. Sommers, *Lusztig's canonical quotient and generalized duality*, J. Algebra **243** (2001), 790–812.

[11] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, Lecture Notes in Math. no. 946, Springer-Verlag, Berlin-New York, 1982.

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803
E-mail address: `pramod@math.lsu.edu`

E-mail address: `sage@math.lsu.edu`